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A Review of Algebra

Mathematics is the language of economics. The only way to model elements of the economy with a level of precision that can garner the precise forecasts necessary for managerial economics is to use algebra and calculus. The material below reviews some concepts from algebra that you should have been exposed to in the past.

Algebra[1]

Algebra, the language of math, uses letters and symbols to express a whole range of problems and solutions. Algebra expresses general relationships that can apply to any set of specific numbers. It's a fundamental tool of business math.

THE BASIC RULES

1. The result of addition is a *sum*.
2. The result of subtraction is a *difference*.
3. The result of multiplication is a *product*.
4. The result of division is a *quotient*.

On paper, we represent multiplication in different ways: sometimes with the symbols (**x**) or (·), and sometimes simply by placing two quantities side by side, implying that they are to be multiplied. We usually represent division with a line.

Multiplying	Dividing
2 x 4 = 8	$\frac{20}{4} = 5$
2(3+1) = 8	
2n = 8	$\frac{n+4}{n} = 4$

Signed numbers follow clear rules. In both multiplication and division, if two numbers have the same sign, then the answer is positive. If the numbers have opposite signs, then the answer is

[1] Portions of this section have been liberally borrowed from *Smart Math for Business: Essentials of Managerial Finance*. New York: Random House, 1997

negative. When numbers are positive, we often do not show signs, as the assumption is that they are positive.

Multiplying	Dividing
(-5) (4) = -20	$\frac{-20}{-4} = 5$
5 (-4) = -20	$\frac{-20}{4} = -5$
-5 (-4) = 20	$\frac{20}{-4} = -5$

Odd <u>powers</u> of negative numbers are negative. Even powers of negative numbers are positive:

Examples:

$(-3)^3 = (-3)\,(-3)\,(-3) = -27$

$(-3)^4 = (-3)\,(-3)\,(-3)\,(-3) = 81$

<u>Parentheses and brackets</u> keep algebraic operations organized. Do the operations within the parentheses first, then work outward. Powers, or exponents, of an expression are performed before products.

Example:

$x = 5 + \{\, 10\,(4 + 2)^2 \}$ What is X? Observe below.

$x = 5 + \{10\,(6)^2\}$

$x = 5 + \{\, 10\,(36) \,\}$

$x = 5 + \{360\}$

$x = 365$

Order of operations: PEMDASFLTR

What is PEMDASFLTR? A type of cigarette? Is it an acronym for "Please excuse my dear Aunt Sally for losing the radishes?"

No! PEMDASFLTR is a handy mnemonic device for remembering the correct order of operations: **P**arentheses. **E**xponents, **M**ultiplication, **D**ivision, **A**ddition, **S**ubtraction, **F**rom **L**eft **T**o **R**ight.

A *term* is a cluster of one or more numbers and variables connected by multiplication or division. An *expression* is one or more terms connected by addition or subtraction. An *equation* is a statement of equality (or inequality) between two algebraic expressions.

Examples:

terms: 2, a, x, 5, x^2

an expression: $2\ (a + 5) + x^2$

an equation: $2\ (a + 5) + x^2 = y$

A *coefficient* is a constant (that is, a number--they're called constants because their values don't change), or a term, before a variable. If there is no coefficient, then the variable's coefficient is assumed to be 1.

Examples:

2x the coefficient of x is 2

y the coefficient of y is 1

Independent and Dependent Variables

An *independent variable* is the variable that drives the relation expressed in a formula or equation. Changes in the independent variable (sometimes called an *exogenous* variable) will produce changes in the rest of the relation. The *dependent variable* is the variable determined by the independent variable. Its value will change in response to changes in the independent variable. Usually, the variable x is designated as the independent variable, while y is usually the dependent variable (sometimes called an *endogenous* variable). Sometimes the equation is written to emphasize the independent x variable, like this:

$f(x) = 2x$. The expression reads, "the function of x equals 2x."

Example (from macroeconomics):

One theory in macroeconomics is that the level of household spending (consumption) in an economy depends upon the level of household income. This theory can be expressed, in general, by the following equation:

$$C = a + bI$$

where	C	is consumption, the dependent variable
	a	is a constant
	b	is a coefficient
	I	is income, the independent variable

Example (from finance):

The capital asset pricing model (CAPM)

This theory is that the rate of return of an asset depends in large part upon the riskiness of the asset's return relative to some hypothetical "risk free" asset. This theory can be expressed, in general, by the following equation

$$R = r_f + \beta\ (r_m - r_f)$$

where	R	is the return on the asset, the dependent variable
	r_f	is the return on the "risk free" asset (independent)
	β	is a coefficient (a constant)
	r_m	is the market average rate of return (independent)

ZERO

Multiplying or dividing by zero is a special case. Multiplying by zero will produce an answer of zero. In other words five times nothing is still nothing. Division involving zero is a little more complicated. When the numerator of a fraction equals zero, then the entire fraction equals zero (zero divided by anything is still zero). If, on the other hand, the denominator of a fraction equals zero, the answer is considered "undefined," because there's no meaningful way to divide a quantity by zero.

Examples:

$\frac{0}{100} = 0$ $\frac{100}{0} =$ *meaningless* (or infinity)

THE COMMUTATIVE AND ASSOCIATIVE LAWS

The commutative and associative laws for addition and multiplication state that the order of addition or multiplication does not matter:

$$a + b + c = 6, \quad \text{or} \quad a + c + b = 6, \quad \text{or} \quad c + b + a = 6$$

$$a \times b \times c = 6, \quad \text{or} \quad a \times c \times b = 6, \quad \text{or} \quad c \times b \times a = 6$$

EXPONENTS, ROOTS, AND LOGARITHMS

A number that is multiplied by itself, like 5 x 5 = 25, is said to be raised to the power of 2, which is written mathematically like this: 5^2. If a number or expression such as x, is raised to the 4th power, it is written as x^4. The 4 in this term is called an exponent. The opposite of a power or exponent is called the root. Most commonly, we encounter the square root, such as the square root of 25:

$$5 = \sqrt{25}$$

However, just as exponents can be of any power, so can roots. This can be shown either as part of the radical sign or as an fractional exponent. The 3rd root of 27 can be written this way:

$$3 = \sqrt[3]{27} \quad \text{or} \quad 3 = 27^{\frac{1}{3}}$$

By the way, there's no need to panic if you encounter a negative exponent, as in x^{-3}, it just means one divided by x^3, that is:

$$\frac{1}{x^3}$$

Terms raised to powers can be multiplied and divided as long as they have the same base. Just follow this rule: Add the exponents when multiplying and subtract them when dividing:

Multiplying	Dividing
$x^3x^2 = x^5$	$\frac{x^5}{x^2} = x^3$

If a term with a power is raised to a power itself, then you multiply the exponents:

$$(x^3)^4 = x^{12}$$

Logarithms, Exponential Functions, and Natural Exponential Functions

Logarithms may seem a little esoteric, but they are really just another way of writing exponential algebra.

Example of logarithms

Base 10: $\log_{10} 1000 = 3$ since $10^3 = 1000$

Base 3: $\log_3 81 = 4$ since $3^4 = 81$

Natural logarithms: Natural logarithms use the base ***e***, a number roughly equal to 2.71828. (Sometimes the "e" is referred to as "exp" on a calculator or spreadsheet.) Natural logarithms have cool uses in business math, especially when one has constant exponential growth (like continuously compounding interest).

Examples of natural logarithms (base e):

$\ln 521 = 6.25575$ since $e^{6.25575} = 521$

$\ln e = 1$ since $e^1 = e$

$\ln e^2 = 2$ since $e^2 = e^2$

Many fancy shmancy calculators can solve for logarithms and natural logarithms. Consult your owner's manual.

Exponential function example: Future value, interest compounded annually

The basic future value equation is:

$$FV = PV\,(1+r)^t$$

where

FV	is the future value of an asset ("t" years from now)
PV	is the present value of the asset
r	is the annual interest rate that the asset earns (simple interest)
t	is the number of years in the future

So, if you deposit \$100 in a bank account that pays 3% compounded *annually*, then how much interest will you earn after 6 years?

$FV = \$100(1+.03)^6 = \$119.40$ so interest income is \$19.40

Example: Natural exponential function, continuously compounding interest

The future value equation for continuous compounding:

$$FV = PV\, e^{rt}$$

where	FV	is the future value of an asset ("t" years from now)
	PV	is the present value of the asset
	e	is the natural base (roughly 2.71828)
	r	is the annual interest rate that the asset earns
	t	is the number of years in the future

So, if you deposit $100 in a bank account that pays 3% compounded *continuously*, then how much interest will you earn after 6 years?

$FV = 100e^{.03(6)} = \$119.72$ so interest earned is $19.72

INEQUALITIES

To show that an expression is greater than another expression, a greater-than (>) or less-than (<) sign is used. If the expressions could also be equal, then we say "greater-than-or-equal-to" or "less-than- or-equal-to" and draw the symbols this way: ≥ and ≤.

POLYNOMIALS AND MORE TERMS

A *monomial* is a single term, such as 3axz.

A *binomial* is an expression having two terms, such as 3axz + 4x.

A *polynomial* is any expression containing two or more terms, such as 3axz + 4x + 6.

A *linear equation* is one that contains no exponents, like y = 2x. (A graph of a linear equation is a straight line.)

A *quadratic equation* is one in which the highest power term is raised to the second power, as in $4x^2 + 3x + 5 = 0$.

Multiplying

When multiplying a monomial, binomial, or polynomial by a certain factor, simply follow the normal multiplication rules, multiplying each term by that factor in turn:

Example: Multiplying (4x + 2) by (x + 3)

$$\begin{array}{rl} 4x + 2 & \\ \underline{x + 3} & \\ & \\ 12x + 6 & \leftarrow \text{multiply the top expression by 3} \\ \underline{4x^2 + 2x \quad} & \leftarrow \text{multiply the top expression by x} \\ 4x^2 + 14x + 6 & \leftarrow \text{add the two resultant expressions} \end{array}$$

Another method to multiply binomials: FOIL. **F**irst, **O**utside, **I**nside, **L**ast

REARRANGING EQUATIONS

Using algebraic rules, we can rearrange an equation to solve for unknown variables. The basic rule for rearranging is that if you add, subtract, multiply, or divide each side of an equation (on different sides of the equal sign) by the same value, the equation will remain correct. (It's kind of like adding or subtracting the same amount of weight on both sides of a scale—it'll stay balanced). You can therefore change an equation any way you like, as long as you change both sides together in the same way

Example: From future value to present value.

Let's rearrange the future value equation to solve for the present value.

We can reverse the equation, putting present value on the left:

$$PV\ (1 + r)^t = FV$$

Then we can divide both sides by the term $(1+r)^t$:

$$\frac{PV(1+r)^t}{(1+r)^t} = \frac{FV}{(1+r)^t}$$

We can then cancel the terms $(1+r)^t$ from the left, and get our present value formula:

$$PV = \frac{FV}{(1+r)^t}$$

If you have a fraction in the denominator, like this:

$$\frac{a}{\frac{c}{b}}$$

You can simplify this term through rearrangement. The rule is to bring the fraction from the bottom to the top of the fraction and flip it vertically, resulting in this simplified term:

$$\frac{a \cdot b}{c}$$

SOLVING EQUATIONS

Solving an equation with one unknown

One solves for a linear equation with one unknown by rearranging the equation to get the unknown by itself on one side of the "=" sign

Example: $3p = 100$

divide both sides by 3: $p = 33.333$ (rounded)

Example: $2x + 10 = 100$

subtract 10 from both sides: $2x = 90$

divide both sides by 2: $x = 45$

Example: $3x = 100 - 2x$

add 2x to both sides: $5x = 100$

divide both sides by 5: $x = 20$

Example: $4x^2 = 100$

divide both sides by 4: $x^2 = 25$

take the square root of 25: $x = 5$

Example: $3(y + 10) = 2y$

use the distributive rule to eliminate parentheses: $3y + 30 = 2y$

subtract 30 from both sides: $3y = 2y - 30$

subtract 2y from both sides: $y = -30$

Example: $10/C = -100$

multiply both sides by C: $10 = -100C$

divide both sides by –100: $-.1 = C$

Solving Quadratic Equations:

Quadratic equations are common in business math. There are several techniques to solve this type of equation for the unknown “x”(note: quadratic equations usually have two answers, though both won't necessarily make sense).

To solve a quadratic equation directly, you can also use the quadratic formula. This is the only way to solve quadratic equations when they don't factor neatly. Simply substitute the values from the standard sequence of the quadratic equation,

$ax^2 + bx + c$, into this formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: A supply equation

Goofy’s corn farm has the following supply equation:

$P - 3Q^2 + 10Q = 5$

where P is price per megabushel
Q is quantity of megabushels produced

At a price of $2 per megabushel, how many megabushels will Goofy produce?

Solving a System of Simultaneous Equations

There are many ways to solve a system of equations for unknown variables. (Note, however, that there must be at least as many equations as there are unknowns.) We shall illustrate 2 methods below: the *subtraction method*, and the *substitution method*.

Example:
Here are the two equations that we will solve (for x and y).

$6x + 2y = 14$
$2x + y = 5$

Method 1: Subtraction method
To solve this system equation by the first method, we multiply the second equation by 2 and get a new second equation:

$6x + 2y = 14$
$\mathbf{4x + 2y = 10}$

Then we subtract the second equation from the first, to get:

$$
\begin{array}{rl}
 & 6x + 2y = 14 \\
- & \underline{4x + 2y = 10} \\
 & \mathbf{2x + \ \ 0 = \ 4}
\end{array}
$$

Then divide both sides of the subtraction by 2:

$\mathbf{x = 2}$

Now that we know the value of x, we can plug it into either one of our original 2 equations to solve for Y. Let's use the first equation,

$6x + 2y = 14$

$6(\mathbf{2}) + 2y = 14$ ←replace the x with its known value, 2

$12 + 2y = 14$ ←multiply 6 times 2

$2y = 2$ ←subtract 12 from both sides

$\mathbf{y = 1}$ ←divide both sides by 2

Method 2: The substitution method:

Step 1: Rearrange the second equation, so that only x is on the left hand side of the "="

$2x + y = 5$ → $2x = 5 - y$ → $2x(.5) = (5-y).5$ → $\mathbf{x = 2.5 - .5y}$

Step 2: Substitute the rearranged second equation into the first equation

$6(2.5 - .5y) + 2y = 14$

Step 3: There ain't no x in the first equation no more, so we can solve it for y

Use distributive law to eliminate parentheses: $15 - 3y + 2y = 14$

Add –3y and 2y: $15 - y = 14$

Add y to both sides: $15 = 14 + y$

Subtract 14 from both sides: $1 = y$

Step 4: Now that we have a value for y, we can use either original equation to solve for x

$6x + 2(1) = 14$ → $6x = 12$ → $x = 2$